

ON THE PRODUCT OF DEDEKIND ZETA FUNCTIONS

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ABSTRACT. We find an expression for the product of two Dedekind zeta functions of a quadratic number field. The main tools used are Riesz sum of order one and the so-called Nakajima dissection.

1. INTRODUCTION

Let K be a number field of degree d with signature (r_1, r_2) , (i.e., $d = r_1 + 2r_2$), h_K denote the class number, D_K the discriminant, R_K regulator and w_K the number of roots of unity in K . If $v_K(m)$ denotes the number of non-zero integral ideals in K with norm m , then the Dedekind zeta-function ζ_K of K is defined by

$$\zeta_K(s) = \sum_{m=1}^{\infty} \frac{v_K(m)}{m^s}, \quad s = \sigma + it$$

for $\sigma > 1$. For any two positive integers m and n , we define

$$\sigma'_{m, v_K}(n) := \sum_{t|n} t^m v_K(t) v_K\left(\frac{n}{t}\right).$$

Riesz means were introduced by M. Riesz [4] and have been studied in connection with summability of Fourier series and of Dirichlet series (for details, we refer [2, 3, 5, 8]). For a given increasing sequence $\{\alpha_n\}$ of real numbers and a given sequence $\{\lambda_n\}$ of complex numbers, the Riesz sum of order κ is defined by

$$\begin{aligned} \mathcal{A}^\kappa(x) = \mathcal{A}_\lambda^\kappa(x) &= \sum'_{\lambda_n \leq x} (x - \lambda_n)^\kappa \alpha_n \\ &= \kappa \int_0^x (x - t)^{\kappa-1} \mathcal{A}_\lambda(t) dt \\ &= \kappa \int_0^x (x - t)^{\kappa-1} d\mathcal{A}_\lambda(t) \end{aligned}$$

with $\mathcal{A}_\lambda(x) = \mathcal{A}_\lambda^0(x) = \sum'_{\lambda_n \leq x} \alpha_n$, where the prime on the summation sign means that when $\lambda_n = x$, the corresponding term is to be halved. Sometimes

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normalized $\frac{1}{\Gamma(\kappa+1)}\mathcal{A}^\kappa(x)$ that appears in the Perron's formula

$$\frac{1}{\Gamma(\kappa+1)}\sum'_{\lambda_n \leq x} \alpha_n(x - \lambda_n)^\kappa = \frac{1}{2\pi i} \int_{(C)} \frac{\Gamma(w)\varphi(w)x^{\kappa+w}}{\Gamma(w+\kappa+1)}dw$$

is also called the Riesz sum of order κ with $\varphi(w) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n^w}$.

We consider the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n^s} \text{ and } \Phi(s) = \sum_{m=1}^{\infty} \frac{a_m}{\gamma_m^s}$$

with $\{\lambda_n\}$ and $\{\gamma_m\}$ are increasing sequences of real numbers. Here α_n and a_m are complex numbers. We assume that these are continued to meromorphic functions over whole complex plane and that they satisfy suitable growth conditions.

Further we consider the following integral,

$$\begin{aligned} \mathcal{F}_{(C)}^\kappa(\varphi(u), \Phi(v); x) &= \frac{1}{2\pi i} \int_{(C)} \frac{\Gamma(w)}{\Gamma(w+\kappa+1)} \varphi(u+w) \Phi(v-w) x^{w+\kappa} dw \\ &= \frac{1}{2\pi i} \sum_{m=1}^{\infty} \frac{a_m}{\gamma_m^v} \int_{(C)} \frac{\Gamma(w)}{\Gamma(w+\kappa+1)} \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n^{u+w}} \gamma_m^w x^{w+\kappa} dw \\ &= \frac{1}{2\pi i} \sum_{m=1}^{\infty} \frac{a_m}{\gamma_m^{v+\kappa}} \int_{(C)} \frac{\Gamma(w)}{\Gamma(w+\kappa+1)} \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n^{u+w}} (\gamma_m x)^{w+\kappa} dw \\ &= \frac{1}{\Gamma(\kappa+1)} \sum_{m=1}^{\infty} \frac{a_m}{\gamma_m^{v+\kappa}} \sum'_{\lambda_n \leq \gamma_m x} \alpha_n \lambda_n^{-u} (\gamma_m x - \lambda_n)^\kappa. \end{aligned}$$

Similarly we have,

$$\mathcal{F}_{(C)}^\kappa(\Phi(v), \varphi(u); x) = \frac{1}{\Gamma(\kappa+1)} \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n^{u+\kappa}} \sum'_{\gamma_m \leq \lambda_n x} a_m \gamma_m^{-v} (\lambda_n x - \gamma_m)^\kappa.$$

Let $\varphi(u) = \zeta_K(u)$ and $\Phi(v) = \zeta_K(v)$ with order $\kappa = 1$. Then by using the above expressions we obtain,

$$\begin{aligned} &\mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); x) + \mathcal{F}_{(C)}^1(\zeta_K(v), \zeta_K(u); x) \\ &= \sum_{m=1}^{\infty} \frac{v_K(m)}{m^{v+1}} \sum'_{n \leq mx} \frac{v_K(n)}{n^u} (mx - n) + \sum_{n=1}^{\infty} \frac{v_K(n)}{n^{u+1}} \sum'_{m \leq nx} \frac{v_K(m)}{m^v} (nx - m). \end{aligned}$$

We now differentiate with respect to x and then substitute $x = 1$ to obtain

$$\begin{aligned} &(\mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); 1))' + (\mathcal{F}_{(C)}^1(\zeta_K(v), \zeta_K(u); 1))' \\ &= \sum_{m=1}^{\infty} \frac{v_K(m)}{m^v} \sum'_{n \leq m} \frac{v_K(n)}{n^u} + \sum_{n=1}^{\infty} \frac{v_K(n)}{n^u} \sum'_{m \leq n} \frac{v_K(m)}{m^v}. \end{aligned}$$

M. Nakajima [6] introduced a dissection (termed as ‘Nakajima dissection’) involving the splitting of double sum

$$\sum_{m,n=1}^{\infty} = \sum_{m=1}^{\infty} \sum'_{n \leq m} + \sum_{n=1}^{\infty} \sum'_{m \leq n}.$$

This dissection technique immediately gives that,

$$\begin{aligned} & (\mathcal{F}_{(C)}^1(\varphi(u), \Phi(v); 1))' + (\mathcal{F}_{(C)}^1(\Phi(v), \varphi(u); 1))' \\ &= \sum_{m=1}^{\infty} \frac{v_K(m)}{m^v} \sum_{n=1}^{\infty} \frac{v_K(n)}{n^u} \\ &= \zeta_K(u) \zeta_K(v). \end{aligned} \tag{1.1}$$

The Meijer G-function is a sum of hypergeometric functions each of which is usually an entire function. It is written as on the left hand side of the following equation and defined [7] by the integral on the right hand side

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_{(C)} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds.$$

The poles of the integrand must be simple and those of $\Gamma(b_j - s)$, $j = 1, \dots, m$, must lie on one side of the contour C and those of $\Gamma(1 - a_j + s)$, $j = 1, \dots, n$, must lie on the other side.

We are now in a position to state the result which is analogous to that of Wilton’s [9]. More precisely, we show:

Theorem 1.1. *Let $\Re(u) > -1$, $\Re(v) > -1$, $\Re(u + v) > 0$ and $u + v \neq 2$. Then for real quadratic fields K ,*

$$\begin{aligned} \zeta_K(u) \zeta_K(v) &= \frac{4h_K R_K}{w_K |D_K|^{1/2}} \left(\frac{1}{u-1} + \frac{1}{v-1} \right) \zeta_K(u+v-1) + 2(2\pi)^{2(u-1)} \\ &\quad |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \left[G_{3,1}^{0,3} \left(\begin{matrix} 1, u, u \\ 0, - \end{matrix} \middle| \frac{D_K}{4\pi^2 m} \right) - \right. \\ &\quad \left. \cos(\pi v) G_{3,1}^{0,3} \left(\begin{matrix} 1, u, u \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right) \right] + 2(2\pi)^{2(v-1)} |D_K|^{\frac{1-2v}{2}} \\ &\quad \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{v-1} \left[G_{3,1}^{0,3} \left(\begin{matrix} 1, v, v \\ 0, - \end{matrix} \middle| \frac{D_K}{4\pi^2 m} \right) - \cos(\pi v) \right. \\ &\quad \left. G_{3,1}^{0,3} \left(\begin{matrix} 1, v, v \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right) \right]. \end{aligned}$$

Also for imaginary quadratic fields K ,

$$\begin{aligned}\zeta_K(u)\zeta_K(v) &= \frac{4h_K R_K}{w_K |D_K|^{1/2}} \left(\frac{1}{u-1} + \frac{1}{v-1} \right) \zeta_K(u+v-1) + 2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \\ &\quad \sin(\pi u) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} G_{3,1}^{0,3} \left(\begin{matrix} 1, u, u \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right) + 2(2\pi)^{2(v-1)} \\ &\quad |D_K|^{\frac{1-2v}{2}} \sin(\pi v) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{v-1} G_{3,1}^{0,3} \left(\begin{matrix} 1, v, v \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right).\end{aligned}$$

Remark 1.1. Theorem 1.1 reduces to Wilton's formula [9] if $K = \mathbb{Q}$. An alternative proof of that formula is recently given by D. Banerjee and J. Mehta [1].

The Riesz sum set up, Perron's formula and Nakajima dissection are the main ingredients in the proof of Theorem 1.1.

2. PROOF OF THE THEOREM 1.1

Let

$$\mathcal{F}_{(-B)}^\kappa(\zeta_K(u), \zeta_K(v); x) = \frac{1}{2\pi i} \int_{(-B)} \frac{\Gamma(w)}{\Gamma(w + \kappa + 1)} \zeta_K(u+w) \zeta_K(v-w) x^{w+\kappa} dw \quad (2.1)$$

with $0 < \Re(u) - 1 < B < \frac{3}{2}$ and $0 < \Re(v) - 1 < B < \frac{3}{2}$. The functional equation satisfied by $\zeta_K(u+w)$ is

$$\begin{aligned}\zeta_K(u+w) &= 2^{-r_2(1-2u-2w)} \pi^{-\frac{d}{2}(1-2u-2w)} |D_K|^{\frac{1-2u-2w}{2}} \frac{\Gamma(\frac{1-u-w}{2})^{r_1}}{\Gamma(\frac{u+w}{2})^{r_1}} \\ &\quad \times \frac{\Gamma(1-u-w)^{r_2}}{\Gamma(u+w)^{r_2}} \zeta_K(1-u-w).\end{aligned} \quad (2.2)$$

Again,

$$\begin{aligned}\frac{\Gamma(\frac{1-u-w}{2})^{r_1}}{\Gamma(\frac{u+w}{2})^{r_1}} &= \frac{[\Gamma(\frac{1-u-w}{2}) \Gamma(1 - \frac{u+w}{2})]^{r_1}}{[\Gamma(\frac{u+w}{2}) \Gamma(1 - \frac{u+w}{2})]^{r_1}} \\ &= \frac{[\Gamma(\frac{1-u-w}{2}) \Gamma(\frac{1}{2} + \frac{1-u-w}{2})]^{r_1}}{\left(\frac{\pi}{\sin \frac{\pi}{2}(u+w)} \right)^{r_1}} \\ &= \left(\frac{2^{u+w}}{\sqrt{\pi}} \right)^{r_1} \left(\sin \frac{\pi}{2}(u+w) \right)^{r_1} \Gamma(1-u-w)^{r_1}.\end{aligned} \quad (2.3)$$

Similarly,

$$\begin{aligned}\frac{\Gamma(1-u-w)^{r_2}}{\Gamma(u+w)^{r_2}} &= \frac{\Gamma(1-u-w)^{2r_2}}{\{\Gamma(u+w) \Gamma(1-u-w)\}^{r_2}} \\ &= \left(\frac{\sin \pi(u+w)}{\pi} \right)^{r_2} \Gamma(1-u-w)^{2r_2}.\end{aligned} \quad (2.4)$$

Using (2.3) and (2.4) in (2.2), we obtain

$$\begin{aligned}\zeta_K(u+w) &= 2^{d(u+w)} \pi^{-d(1-u-w)} |D_K|^{\frac{1-2u-2w}{2}} \left\{ \sin \frac{\pi}{2}(u+w) \right\}^{r_1+r_2} \\ &\times \left\{ \cos \frac{\pi}{2}(u+w) \right\}^{r_2} \Gamma(1-u-w)^d \zeta_K(1-u-w).\end{aligned}$$

Thus (2.1) becomes

$$\begin{aligned}\mathcal{F}_{(-B)}^\kappa(\zeta_K(u), \zeta_K(v); x) &= \frac{1}{2\pi i} \int_{(-B)} S_\kappa(w) f(w) 2^{d(u+w)} \pi^{-d(1-u-w)} \\ &\times |D_K|^{\frac{1-2u-2w}{2}} \left\{ \sin \frac{\pi}{2}(u+w) \right\}^{r_1+r_2} \left\{ \cos \frac{\pi}{2}(u+w) \right\}^{r_2} \\ &\times \Gamma(1-u-w)^d x^{w+\kappa} dw.\end{aligned}\tag{2.5}$$

Here $S_\kappa(w) = \frac{\Gamma(w)}{\Gamma(w+\kappa+1)}$ and $f(w) = \zeta_K(1-u-w)\zeta_K(v-w)$.

We now consider K to be a quadratic field. Then (2.5) becomes

$$\begin{aligned}\mathcal{F}_{(-B)}^\kappa(\zeta_K(u), \zeta_K(v); x) &= \frac{1}{2\pi i} \int_{(-B)} S_\kappa(w) f(w) 2^{2(u+w)} \pi^{-2(1-u-w)} \\ &\times |D_K|^{\frac{1-2u-2w}{2}} \left\{ \sin \frac{\pi}{2}(u+w) \right\}^{r_1+r_2} \left\{ \cos \frac{\pi}{2}(u+w) \right\}^{r_2} \\ &\times \Gamma(1-u-w)^2 x^{w+\kappa} dw.\end{aligned}$$

Employing change of variables and assuming that $\Re(u) < B$,

$$\begin{aligned}\mathcal{F}_{(-B)}^\kappa(\zeta_K(u), \zeta_K(v); x) &= \frac{2}{\pi i} \int_{(B)} S_\kappa(-z) f(-z) (2\pi)^{-2(1-u+z)} \\ &\times |D_K|^{\frac{1-2u+2z}{2}} \left\{ \sin \frac{\pi}{2}(u-z) \right\}^{r_1+r_2} \left\{ \cos \frac{\pi}{2}(u-z) \right\}^{r_2} \\ &\times \Gamma(1-u+z)^2 x^{-z+\kappa} dz.\end{aligned}\tag{2.6}$$

Again,

$$\begin{aligned}f(-z) &= \zeta_K(1-u+z)\zeta_K(v+z) \\ &= \sum_{m=1}^{\infty} \frac{v_K(m)}{m^{1-u+z}} \sum_{m=1}^{\infty} \frac{v_K(m)}{m^{v+z}} \\ &= \sum_{m=1}^{\infty} \frac{\sum_{d|m} d^{-v} \frac{m^{u-1}}{d^{u-1}} v_K(d) v_K(\frac{m}{d})}{m^z} \\ &= \sum_{m=1}^{\infty} \frac{\sum_{d|m} d^{1-u-v} v_K(d) v_K(\frac{m}{d})}{m^{1-u+z}} \\ &= \sum_{m=1}^{\infty} \frac{\sigma'_{1-u-v, v_K}(m)}{m^{1-u+z}}.\end{aligned}$$

Therefore (2.6) implies that

$$\begin{aligned}\mathcal{F}_{(-B)}^\kappa(\zeta_K(u), \zeta_K(v); x) &= \frac{2}{\pi i} (2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\ &\times \int_{(B)} S_\kappa(-z) \left(\frac{|D_K|}{4\pi^2 m} \right)^z \left\{ \sin \frac{\pi}{2} (u-z) \right\}^{r_1+r_2} \\ &\times \left\{ \cos \frac{\pi}{2} (u-z) \right\}^{r_2} \Gamma(1-u+z)^2 x^{-z+\kappa} dz.\end{aligned}$$

If we now take $\kappa = 1$, then

$$\begin{aligned}\mathcal{F}_{(-B)}^1(\zeta_K(u), \zeta_K(v); x) &= \frac{2}{\pi i} (2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\ &\times \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z \left\{ \sin \frac{\pi}{2} (u-z) \right\}^{r_1+r_2} \\ &\times \left\{ \cos \frac{\pi}{2} (u-z) \right\}^{r_2} \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz.\end{aligned}\tag{2.7}$$

We treat real and imaginary cases separately.

Case-I: Real quadratic fields:

In this case $r_1 = 2$ and $r_2 = 0$. Thus (2.7) gives

$$\begin{aligned}\mathcal{F}_{(-B)}^1(\zeta_K(u), \zeta_K(v); x) &= -\frac{1}{2\pi i} (2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\ &\times \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z \left(e^{i\pi(u-z)} + e^{-i\pi(u-z)} - 2 \right) \\ &\times \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz.\end{aligned}\tag{2.8}$$

Let

$$H_B(u; x) = \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z e^{i\pi(u-z)} \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz.$$

We differentiate $H_B(u; x)$ with respect to x and get,

$$\begin{aligned}H'_B(u; x) &= -e^{i\pi u} \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m x e^{i\pi}} \right)^z \frac{\Gamma(1-u+z)^2}{z} dz \\ &= -e^{i\pi u} \int_{(B)} \left(\frac{-|D_K|}{4\pi^2 m x} \right)^z \frac{\Gamma(z) \Gamma(1-u+z) \Gamma(1-u+z)}{\Gamma(z+1)} dz \\ &= -2\pi i e^{i\pi u} G_{3,1}^{0,3} \left(1, u, u \mid \frac{-|D_K|}{4\pi^2 m x} \right).\end{aligned}$$

Also let,

$$\mathcal{H}_B(u; x) = \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z e^{-i\pi(u-z)} \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz.$$

As before differentiating with respect to x gives

$$\begin{aligned} \mathcal{H}'_B(u; x) &= -e^{-i\pi u} \int_{(B)} \left(\frac{|D_K| e^{i\pi}}{4\pi^2 m x} \right)^z \frac{\Gamma(1-u+z)^2}{z} dz \\ &= -e^{-i\pi u} \int_{(B)} \left(\frac{-|D_K|}{4\pi^2 m x} \right)^z \frac{\Gamma(z) \Gamma(1-u+z) \Gamma(1-u+z)}{\Gamma(z+1)} dz \\ &= -2\pi i e^{-i\pi u} G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m x} \right). \end{aligned}$$

Further, let

$$\mathfrak{H}_B(u; x) = 2 \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz$$

As before,

$$\begin{aligned} \mathfrak{H}'_B(u; x) &= -2 \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m x} \right)^z \frac{\Gamma(1-u+z)^2}{z} dz \\ &= -2 \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m x} \right)^z \frac{\Gamma(z) \Gamma(1-u+z) \Gamma(1-u+z)}{\Gamma(z+1)} dz \\ &= -2\pi i G_{3,1}^{0,3} \left(1, u, u \mid \frac{D_K}{4\pi^2 m x} \right). \end{aligned}$$

Thus we have

$$\begin{aligned} H'_B(u; x) + \mathcal{H}'_B(u; x) + \mathfrak{H}'_B(u; x) &= -4\pi i \cos(\pi u) G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m x} \right) + \\ &\quad 2\pi i G_{3,1}^{0,3} \left(1, u, u \mid \frac{D_K}{4\pi^2 m x} \right). \end{aligned}$$

Therefore (2.8) becomes

$$\begin{aligned} (\mathcal{F}_{(-B)}^1(\zeta_K(u), \zeta_K(v); x))' &= 2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\ &\quad \times \left\{ \cos(\pi u) G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m x} \right) - G_{3,1}^{0,3} \left(1, u, u \mid \frac{D_K}{4\pi^2 m x} \right) \right\}. \end{aligned} \quad (2.9)$$

Clearly this series is absolutely convergent and using Cauchy's residue theorem,

$$\begin{aligned} \mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); x) &= \mathcal{F}_{(-B)}^1(\zeta_K(u), \zeta_K(v); x) + x \zeta_K(u) \zeta_K(v) - \\ &\quad \zeta_K(u-1) \zeta_K(v-1) + \frac{\zeta_K(u+v-1) x^{2-u}}{(u-2)(u-1)} \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}}. \end{aligned}$$

Further differentiating with respect to x gives,

$$(\mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); x))' = (\mathcal{F}_{(-B)}^1(\zeta_K(u), \zeta_K(v); x))' + \zeta_K(u)\zeta_K(v) - \frac{\zeta_K(u+v-1)x^{1-u}}{u-1} \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}}. \quad (2.10)$$

We thus have (using (2.9))

$$\begin{aligned} (\mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); x))' &= 2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\ &\quad \left\{ \cos(\pi u) G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m x} \right) - G_{3,1}^{0,3} \left(1, u, u \mid \frac{D_K}{4\pi^2 m x} \right) \right\} + \zeta_K(u)\zeta_K(v) - \frac{\zeta_K(u+v-1)x^{1-u}}{u-1} \\ &\quad \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}}. \end{aligned}$$

If we substitute $x = 1$,

$$\begin{aligned} (\mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); 1))' &= 2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\ &\quad \left\{ \cos(\pi u) G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m} \right) - G_{3,1}^{0,3} \left(1, u, u \mid \frac{D_K}{4\pi^2 m} \right) \right\} + \zeta_K(u)\zeta_K(v) - \frac{\zeta_K(u+v-1)}{u-1} \\ &\quad \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}}. \quad (2.11) \end{aligned}$$

Similarly,

$$\begin{aligned} (\mathcal{F}_{(C)}^1(\zeta_K(v), \zeta_K(u); x))' &= 2(2\pi)^{2(v-1)} |D_K|^{\frac{1-2v}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{v-1} \\ &\quad \left\{ \cos(\pi v) G_{3,1}^{0,3} \left(1, v, v \mid \frac{-D_K}{4\pi^2 m} \right) - G_{3,1}^{0,3} \left(1, v, v \mid \frac{D_K}{4\pi^2 m} \right) \right\} + \zeta_K(u)\zeta_K(v) - \frac{\zeta_K(u+v-1)}{v-1} \\ &\quad \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}}. \quad (2.12) \end{aligned}$$

Now we add (2.11) and (2.12), and further utilising (1.1) arrive at

$$\begin{aligned}
\zeta_K(u)\zeta_K(v) &= 2\zeta_K(u)\zeta_K(v) - \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}} \left(\frac{1}{u-1} + \frac{1}{v-1} \right) \zeta_K(u+v-1) \\
&\quad + 2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \left\{ \cos \pi u \right. \\
&\quad \left. G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m} \right) - G_{3,1}^{0,3} \left(1, u, u \mid \frac{D_K}{4\pi^2 m} \right) \right\} + 2(2\pi)^{2(v-1)} \\
&\quad |D_K|^{\frac{1-2v}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{v-1} \left\{ \cos(\pi v) G_{3,1}^{0,3} \left(1, v, v \mid \frac{-D_K}{4\pi^2 m} \right) - \right. \\
&\quad \left. G_{3,1}^{0,3} \left(1, v, v \mid \frac{D_K}{4\pi^2 m} \right) \right\}.
\end{aligned}$$

Which implies that,

$$\begin{aligned}
\zeta_K(u)\zeta_K(v) &= \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}} \left(\frac{1}{u-1} + \frac{1}{v-1} \right) \zeta_K(u+v-1) + 2(2\pi)^{2(u-1)} \\
&\quad |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \left\{ G_{3,1}^{0,3} \left(1, u, u \mid \frac{D_K}{4\pi^2 m} \right) - \right. \\
&\quad \left. \cos(\pi u) G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m} \right) \right\} + 2(2\pi)^{2(v-1)} |D_K|^{\frac{1-2v}{2}} \\
&\quad \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{v-1} \left\{ G_{3,1}^{0,3} \left(1, v, v \mid \frac{D_K}{4\pi^2 m} \right) - \cos(\pi v) \right. \\
&\quad \left. G_{3,1}^{0,3} \left(1, v, v \mid \frac{-D_K}{4\pi^2 m} \right) \right\}.
\end{aligned}$$

This concludes the proof of this case.

Case-II: Imaginary quadratic fields:

In this case $r_1 = 0$ and $r_2 = 1$. Thus (2.7) implies that,

$$\begin{aligned}
\mathcal{F}_{(-B)}^1(\zeta_K(u), \zeta_K(v); x) &= \frac{1}{\pi i} (2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\
&\quad \times \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z \sin \pi(u-z) \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz \\
&= -\frac{1}{2\pi} (2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\
&\quad (I_B(u; x) - \mathcal{I}(u; x)). \tag{2.13}
\end{aligned}$$

Where

$$I_B(u; x) = \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z e^{i\pi(u-z)} \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz,$$

and

$$\mathcal{I}_B(u; x) = \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m} \right)^z e^{-i\pi(u-z)} \Gamma(1-u+z)^2 \frac{x^{-z+1}}{z(z-1)} dz.$$

Proceeding analogously we differentiate $I_B(u; x)$ with respect to x and get,

$$\begin{aligned} I'_B(u; x) &= -e^{i\pi u} \int_{(B)} \left(\frac{|D_K|}{4\pi^2 m x e^{i\pi}} \right)^z \frac{\Gamma(1-u+z)^2}{z} dz \\ &= -e^{i\pi u} \int_{(B)} \left(\frac{-|D_K|}{4\pi^2 m x} \right)^z \frac{\Gamma(z) \Gamma(1-u+z) \Gamma(1-u+z)}{\Gamma(z+1)} dz \\ &= -2\pi i e^{i\pi u} G_{3,1}^{0,3} \left(1, u, u \mid \frac{-|D_K|}{4\pi^2 m x} \right). \end{aligned}$$

Again differentiating $\mathcal{I}_B(u; x)$ with respect to x , we get

$$\begin{aligned} \mathcal{I}'_B(u; x) &= -e^{-i\pi u} \int_{(B)} \left(\frac{-|D_K|}{4\pi^2 m x} \right)^z \frac{\Gamma(1-u+z)^2}{z} dz \\ &= -2\pi i e^{-i\pi u} G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m x} \right). \end{aligned}$$

Thus (2.13) becomes

$$\begin{aligned} (\mathcal{F}_{(-B)}^1(\zeta_K(u), \zeta_K(v); x))' &= -2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sin(\pi u) \sigma'_{1-u-v, v_K}(m) \\ &\quad m^{u-1} G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m x} \right). \end{aligned}$$

We now use (2.10),

$$\begin{aligned} (\mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); x))' &= -2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sin(\pi u) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) \\ &\quad m^{u-1} G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m x} \right) + \zeta_K(u) \zeta_K(v) - \\ &\quad \frac{\zeta_K(u+v-1) x^{1-u}}{u-1} \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}}. \end{aligned}$$

Further substituting $x = 1$,

$$\begin{aligned} (\mathcal{F}_{(C)}^1(\zeta_K(u), \zeta_K(v); x))' &= -2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sin(\pi u) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) \\ &\quad m^{u-1} G_{3,1}^{0,3} \left(1, u, u \mid \frac{-D_K}{4\pi^2 m} \right) + \zeta_K(u) \zeta_K(v) - \\ &\quad \frac{\zeta_K(u+v-1)}{u-1} \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}} \end{aligned} \tag{2.14}$$

Similarly we obtain

$$\begin{aligned}
(\mathcal{F}_{(C)}^1(\zeta_K(v), \zeta_K(u); x))' &= -2(2\pi)^{2(v-1)} |D_K|^{\frac{1-2v}{2}} \sin(\pi v) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) \\
&\quad m^{v-1} G_{3,1}^{0,3} \left(\begin{matrix} 1, v, v \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right) + \zeta_K(u) \zeta_K(v) - \\
&\quad \frac{\zeta_K(u+v-1)}{v-1} \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}}. \tag{2.15}
\end{aligned}$$

Adding (2.14) and (2.15), and then applying (1.1) we obtain,

$$\begin{aligned}
\zeta_K(u) \zeta_K(v) &= 2\zeta_K(u) \zeta_K(v) - \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}} \left(\frac{1}{u-1} + \frac{1}{v-1} \right) \zeta_K(u+v-1) \\
&\quad - 2(2\pi)^{2(u-1)} |D_K|^{\frac{1-2u}{2}} \sin(\pi u) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} \\
&\quad G_{3,1}^{0,3} \left(\begin{matrix} 1, u, u \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right) - 2(2\pi)^{2(v-1)} |D_K|^{\frac{1-2v}{2}} \sin(\pi v) \\
&\quad \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{v-1} G_{3,1}^{0,3} \left(\begin{matrix} 1, v, v \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right).
\end{aligned}$$

Finally this gives the required result:

$$\begin{aligned}
\zeta_K(u) \zeta_K(v) &= \frac{4h_K R_K}{w_K |D_K|^{\frac{1}{2}}} \left(\frac{1}{u-1} + \frac{1}{v-1} \right) \zeta_K(u+v-1) + 2(2\pi)^{2(u-1)} \\
&\quad |D_K|^{\frac{1-2u}{2}} \sin(\pi u) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{u-1} G_{3,1}^{0,3} \left(\begin{matrix} 1, u, u \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right) + \\
&\quad 2(2\pi)^{2(v-1)} |D_K|^{\frac{1-2v}{2}} \sin(\pi v) \sum_{m=1}^{\infty} \sigma'_{1-u-v, v_K}(m) m^{v-1} \\
&\quad G_{3,1}^{0,3} \left(\begin{matrix} 1, v, v \\ 0, - \end{matrix} \middle| \frac{-D_K}{4\pi^2 m} \right).
\end{aligned}$$

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